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Gravity and Gauge: A New Perspective

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Realization of the Poincaré group P_{10} as a subgroup of $GL(5, \mathbb{R})$ that maps a 4-dimensional affine set into itself has been shown to lead to a direct Yang-Mills gauging process. This paper discusses the differences between direct gauge theory for P_{10} and previously published works. These differences are fundamental, both physically and mathematically, and lead to marked departures from previous concepts and interpretations. The translation subgroup is correctly gauged; the metric structure and metric compatibility are derived from the gauging process rather than assumed; spin structures are automatically incorporated in a consistent manner; the local holonomy group is shown to be the component of the Lorentz group connected to the identity; the geometric analog of Yang-Mills minimal coupling precludes dependence of the free gauge field Lagranian on torsion; and the theory reduces exactly to general relativity when the momentum-energy complex is symmetric and all matter fields are spin-free. Gravitational effects on neutral test particles are shown to arise from the compensating 1-forms for local action of Lorentz boosts. The compensating 1-forms for local action of the translation subgroup may be interpreted as space-time dislocations, while the compensating 1-forms for the rotation subgroup can be viewed as space-time disclinations. Unfortunately, there are no clear physical meanings that can be ascribed to space-time dislocations or disclinations.

Gauge theories based on the Poincaré group started with the fundamental paper by Utiyama (1956), although Utiyama's treatment was more nearly a gauge theory for the Lorentz group. The paper by Kibble (1961) seems to be the first serious attempt to bring the translation subgroup of the Poincaré group into the gauging process on an equal footing. This was achieved by introducing a system of fundamental frame and coframe fields and then identifying the coframe fields with the compensating fields for local action of the translation group. The natural geometry associated with

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the coframe fields led to differentiable manifolds with nontrivial curvature and torsion. The geometry of Kibble's theory and in those of Hehl *et al.* (1976, 1980) and Dreschsler (1982) was thus put in first, with the physics following as an overlay introduced by identifying some of the geometric fields with appropriate physical fields and the demand that the coframe fields be gauge covariant constant. This latter demand fixed the connection coefficients in a manner that guaranteed satisfaction of the Ricci lemma for the induced "metric tensor." The underlying differentiable manifold thus became a Riemann-Cartan space, but this result was obtained by an additional assumption rather than as a consequence of the underlying gauge constructs.

The intrinsic difficulties in gauging the Poincaré group arise because the Poincaré group is not a semisimple group, and because it acts both on the matter fields and on the underlying space-time (base) maifold. Although the lack of semisimplicity leads to a singular Cartan-Killing form, and hence to difficulties in constructing an appropriate Lagrangian function for the compensating fields, this aspect of the difficulty is relatively easy to overcome. On the other hand, the additive action of the translation group on the base manifold has been a consistent stumbling block. As Schweizer (1980) puts it: "So far nobody has managed to gauge the translation group in a satisfactory manner." Similar conclusions, although less didactically stated, are conveyed in the articles by Basombrio (1980) and Trautman (1982). Alternative approaches have been discussed by Wess (1983), Aldrovandi and Stedile (1984), and in the papers reported in Kikkawa *et al.* (1982), but a consensus is yet to emerge as to the applicability of the Poincaré gauge theory to the fundamental unification problem. In fact, this lack of consensus is one of the reasons for the extensive current investments in string theory and supersymmetry-supergravity.

A fundamentally different alternative grew out of a paper on operator-valued connections and gauge constructions for arbitrary, finite parameter Lie groups (Edelen, 1984). An equivalent, but simpler, direct physical theory for the Poincaré group has been reported in Edelen (1985*a*–*d*, 1986*a*), and the geometry of the resulting space-time has been analyzed in Edelen (1986*b*). This is a direct gauge theory in which all fundamental quantities grow out of Minkowski space-time by the action of a well-defined minimial replacement operator. The physics is put in first, and the geometry and mathematics then take care of themselves. A summary of the results is given in this paper, where I will concentrate on pointing out the differences between direct gauge theory for the Poincaré group and other approaches that have been reported in the literature. The results are such that the view of Schweizer quoted above can be put to rest; that is, the translation group has now been gauged in a satisfactory manner.

1. MINIMAL REPLACEMENT FOR THE POINCARÉ GROUP

The theory starts with a collection of matter fields Ψ (indices suppressed) on Minkowski space-time M_4. Since M_4 is both curvature-free and torsion-free, there is no loss of generality in introducing a specific global coordinate cover $\{x^i | 1 \le i \le 4\}$ and a specific system of units relative to which the metric tensor h_{ij} of M_4 takes the diagonal form diag$(1, 1, 1, -1)$.

The dynamics of the matter fields is described by a Lagrangian function $L_0(\Psi, \partial_i \Psi)$ that is invariant under an r-parameter group G_r of internal symmetries and under the Poincaré group P_{10}. If the action of the Poincaré group is kept global, while the internal symmetry group G_r is allowed to act locally, standard Yang-Mills gauge theory starting with L_0 is assumed to provide a correct description of the physics on Minkowski space-time (i.e., with gravity "switched off"). On the other hand, the goal of Poincaré theory is to "switch on" gravity by gauging the total group $G_r \oplus P_{10}$. The direct product structure of the total group shows that we need to study the gauge-theoretic problem associated with allowing the Poincaré group to act locally.

The results reported by Aldrovandi and Stedile (1984) show that the Poincaré group can be gauged as a matrix Lie group in the standard Yang-Mills fashion by realizing the Poincaré group as a subgroup of $GL(5, \mathbb{R})$ that maps an affine set into itself. This representation serves to define the minimal replacement operator $\mathcal{M}$ and the fundamental coframe fields (Edelen, 1984, 1986*b*)

$$\mathcal{M}\langle dx^i \rangle = B^i = (\delta^i_j + W^\alpha_j l^i_{\alpha k} x^k + \phi^i_j)\, dx^j \tag{1}$$

where

$$W^\alpha = W^\alpha_j\, dx^j, \qquad 1 \le \alpha \le 6 \tag{2}$$

are the compensating 1-forms for the Lorentz sector $L(4, \mathbb{R})$,

$$\phi^i = \phi^i_j\, dx^j, \qquad 1 \le i \le 4 \tag{3}$$

are the compensating 1-forms for the translation sector $T(4)$, and

$$l^i_{\alpha k}, \qquad 1 \le \alpha \le 6, \quad 1 \le i, k \le 4$$

are the components of a matrix basis for the matrix Lie algebra of the Lorentz group.

The *distortion* 1-forms $\{B^i | 1 \le i \le 4\}$ form a basis for the vector space of 1-form provided

$$B^1 \wedge B^2 \wedge B^3 \wedge B^4 = B\mu \ne 0 \tag{4}$$

where

$$B = \det(B^i_j), \qquad \mu = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \tag{5}$$

The dual basis $\{b_i | 1 \le i \le 4\}$, defined by

$$b_i \lrcorner B^j = \delta_i^j, \qquad b_i = b_i^k \partial_k \tag{6}$$

gives

$$\mathcal{M}\langle \partial_i \rangle = b_i \tag{7}$$

Thus, if we define $\mathcal{M}\langle f \rangle = f$ for scalar-valued functions of position on M_4, minimal replacement extends to the full tensor algebra on Minkowski space-time by

$$\mathcal{M}\langle A + B \rangle = \mathcal{M}\langle A \rangle + \mathcal{M}\langle B \rangle, \qquad \mathcal{M}\langle A \otimes B \rangle = \mathcal{M}\langle A \rangle \otimes \mathcal{M}\langle B \rangle \tag{8}$$

Application of the minimal replacement operator may be envisioned as lifting Minkowski space-time and its geometric object fields up to a new space time L_4 and its geometric object fields, where M_4 and L_4 use the same local coordinate functions:

$$\mathcal{M}\langle M_4 \rangle = L_4 \tag{9}$$

$$\mathcal{M}\langle v^j \partial_j \rangle = v^j b_j = v^j b_j^i \partial_i = V \in T(L_4) \tag{10}$$

$$\mathcal{M}\langle a_j \, dx^j \rangle = a_j B^j = a_j B_i^j \, dx^i = A \in \Lambda^1(L_4) \tag{11}$$

Components of geometric object fields on M_4 thus lift to corresponding components of geometric object fields on L_4, but referred to anholonomic bases. Application of the minimal replacement operator to the metric form on M_4 gives

$$dS^2 = \mathcal{M}\langle h_{ij} \, dx^i \otimes dx^j \rangle = g_{ij} \, dx^i \otimes dx^j \tag{12}$$

with

$$g_{ij} = B_i^r h_{rs} B_j^s, \qquad B = \det(B_i^r) = (-g)^{1/2} \tag{13}$$

The second-order covariant tensor field with components g_{ij} defined by (13) is thus the natural candidate for the metric tensor field on the new space-time L_4.

Most previously reported gauge theories for the Poincaré group make an *ad hoc* identification of the fundamental coframe fields with the compensating 1-forms for the translation sector; that is, with the 1-forms $\phi^i = \phi_j^i \, dx^j$. In contrast, direct gauge theory for the Poincaré groups gives explicit representations for the compensating 1-forms for the translation sector and altogether different 1-forms for the coframe fields; namely, the distortion 1-forms

$$B^i = dx^i + W^\alpha l_{\alpha j}^i x^j + \phi^i$$

The coframe fields of direct gauge theory thus contain contributions from the Lorentz sector and the 1-forms dx^i in addition to the compensating 1-forms for the translation sector. This difference has strong and important ramifications. For instance, the metric tensor g_{ij} given by (13) in direct gauge theory is very different from the geometric object field with components

$$\hat{g}_{ij} = \phi^r_i h_{rs} \phi^s_j \tag{14}$$

that has been used for the metric tensor in previous theories (Kibble, 1961; Hehl *et al.*, 1976, 1980; Drechsler, 1982). Although it has been ignored by previous authors, the geometric object field with components given by (14) does not transform as a tensor field on L_4 under transformations induced by local action of the Poincaré group, as we shall see shortly.

2. TRANSFORMATIONS INDUCED BY LOCAL ACTION OF THE POINCARÉ GROUP

Local action of the Poincaré group results in transformations of the local coordinate covers of L_4 of the form

$$'x^i = L^i_j(x^k)x^j + t^i(x^k) \tag{15}$$

where $L^i_j(x^k)$ are the components of a position-dependent Lorentz transformation matrix and $t^i(x^k)$ are position-dependent translation functions (Edelen, 1986*b*). The space L_4 is thus subject to arbitrary smooth changes of coordinate covers as a consequence of the local action of the Poincaré group as a group of point transformations on M^4. Although (15) can be written in the equivalent form $'x^i = f^i(x^k)$, the position-dependent Lorentz matrices remain in the theory because the construction of the frame and coframe fields by minimal replacement gives the transformation laws

$$'B^i = L^i_j(x^k)B^j, \qquad 'b_i L^i_j(x^k) = b_j \tag{16}$$

that is,

$$'B^i_m \frac{\partial' x^m}{\partial x^n} = L^i_j(x^k)B^j_n, \qquad 'b^n_i L^i_j(x^k) = b^m_j \frac{\partial' x^n}{\partial x^m} \tag{17}$$

Local transformations of the Poincaré group thus induce more complicated transformation laws for the frame and coframe fields on L_4 than one customarily finds. In particular, the frame and coframe fields have corresponding "tensor" laws of transformations only when the position-dependent Lorentz tranformation matrices reduce globally to the identity. A careful examination of (17) shows that the lower index on the B's and the upper index on the b's are ordinary tensor indices, while the upper index on the B's and the lower index on the b's are anholonomic indices that respond to the position-dependent Lorentz transformation matrices. Now, it is an easy matter to use (17) in order to show that the metric tensor g_{ij} that is defined on L_4 by (13) has the ordinary tensor law of transformation under local action of the Poincaré group. On the other hand, the geometric object field $\hat{g}_{ij}$ defined by (14) does not transform as a tensor [see (32) below]. Indeed, previous theories that use $\hat{g}_{ij}$ as a metric tensor run into serious difficulties.

The minimal replacement operator induces a *Gauge-covariant exterior derivative* (Edelen, 1985*a*, 1986*b*)

$$DB^i = dB^i + \gamma^i_j \wedge B^j = \Sigma^i \tag{18}$$

that serves to define the associated *Cartan torsion* 2-forms Σ^i, and the anholonomic connection 1-forms

$$\gamma^i_j = W^\alpha l^i_{\alpha j} \tag{19}$$

for the Lorentz sector that take their values in the matrix Lie algebra of the Lorentz group. Converting the anholonomic connection 1-forms to holonomic ones in the standard fashion gives us the ordinary connection 1-forms,

$$\Gamma^i_j = W^\alpha L^i_{\alpha j} + b^i_k \, dB^k_j = \Gamma^i_{mj} \, dx^m \tag{20}$$

for L_4. The quantities

$$L^i_{\alpha j} = b^i_k l^k_{\alpha m} B^m_j \tag{21}$$

define a basis for the Lie algebra of the Lorentz group lifted to L_4; that is,

$$[L_\alpha, L_\beta] = C_\alpha{}^\nu{}_\beta L_\nu, \qquad L^k_{\alpha i} g_{kj} + L^k_{\alpha j} g_{ki} = 0 \tag{22}$$

An easy calculation shows that the Latin indices on the L's transform according to the indicated tensor law, while the Greek indices transform according to the adjoint representation of $L(4, \mathbb{R})$,

$$'L^i_{\alpha j} \frac{\partial' x^j}{\partial x^m} = H^\beta_\alpha L^r_{\beta m} \frac{\partial' x^i}{\partial x^r}, \qquad \mathbf{L} \mathbf{L}_\alpha \mathbf{L}^{-1} = H^\beta_\alpha \mathbf{L}_\beta \tag{23}$$

where $\mathbf{L}$ stands for the local Lorentz matrix of the transformation. The relations (20) show that the connection 1-forms Γ^i_j take their values in the Lie algebra of $L(4, \mathbb{R})$ only when $dB^k_j = 0$, in sharp contrast with γ^i_j and with the usual state of affairs in Yang-Mills theory. Corresponding connection 1-forms for the adjoint representation of $L(4, \mathbb{R})$ are given by (Edelen, 1986*b*)

$$\Gamma^\alpha_\beta = W^\nu C_\nu{}^\alpha{}_\beta \tag{24}$$

These connection 1-forms must be used when computing gauge-covariant derivatives of quantities with Greek indices (for quantities that live in the Lie algebra of the Lorentz group).

The Cartan torsion 2-forms of the γ-connection have been defined in (18). They have the explicit evaluations

$$\Sigma^i = \theta^\alpha l^i_{\alpha j} x^j + d\phi^i + \gamma^i_j \wedge \phi^j \tag{25}$$

while the components of the corresponding homolomic torsion tensor are given by

$$S^k_{ij} = \Gamma^k_{[ij]} = b^k_m \Sigma^m_{ij}, \qquad \Sigma^m = \tfrac{1}{2}\Sigma^m_{ij}\, dx^i \wedge dx^j \tag{26}$$

The reader should note the explicit dependence of the Cartan torsion on the x's that is shown by (25). It will turn out to have particular importance when we come to the problem of constructing "free-field" Lagrangian functions.

The six curvature 2-forms θ^α of the Lorentz sector that appears in (25) have the evaluations

$$\theta^\alpha = dW^\alpha + \tfrac{1}{2} C_\rho{}^\alpha{}_\nu W^\rho \wedge W^\nu = \tfrac{1}{2}\theta^\alpha_{ij}\, dx^i \wedge dx^j \tag{27}$$

The corresponding holonomic curvature tensor for L_4 can be shown to be determined by (Edelen, 1986*b*)

$$R^i_{rsj} = \theta^\alpha_{rs} L^i_{\alpha j} \tag{28}$$

The holonomic curvature 2-forms

$$R^i_j = \tfrac{1}{2} R^i_{rsj}\, dx^r \wedge dx^s = \theta^\alpha L^i_{\alpha j} \tag{29}$$

thus take their values in the Lie algebra of the Lorentz group that is lifted to L_4 by minimal replacement, in sharp contrast with the Γ's.

Translational torsion 2-forms can be defined by (Hehl *et al.*, 1976)

$$\Omega^i = d\phi^i + \gamma^i_j \wedge \phi^j \tag{30}$$

in which case the Cartan torsion becomes

$$\Sigma^i = \theta^\alpha l^i_{\alpha j} x^j + \Omega^i \tag{31}$$

The translational torsion 2-forms Ω^i are what have been used in most previous theories. There is a major difficulty associated with the use of Ω^i as torsion 2-forms and with ϕ^i as coframe fields, however. It can be shown (Edelen, 1985*a*, 1986*b*) that local action of the Poincaré group induces the transformations

$$'\phi^i = L^i_j(x^k)\phi^j - {}'\gamma^i_j t^j(x^k) - dt^i(x^k) \tag{32}$$

$$'\Omega^i = L^i_j(x^k)\Omega^j - '\theta^\alpha l^i_{\alpha j} t^j(x^k) \tag{33}$$

$$'\gamma^i_j = L^i_m(x^k)\gamma^m_n \overset{-1}{L}{}^n_j(x^k) - (dL^i_m(x^k)) \overset{-1}{L}{}^m_j(x^k) \tag{34}$$

$$'\theta^\alpha l^i_{\alpha j} = \theta^\beta L^i_m(x^k) l^m_{\beta n} \overset{-1}{L}{}^n_j(x^k) \tag{35}$$

The ϕ's and the Ω's thus have *affine* rather than linear transformation laws, and this engenders serious problems. In contrast, the B's and the Σ's have the linear transformation laws

$$B^i = L^i_j(x^k)B^j, \qquad '\Sigma^i = L^i_j(x^k)\Sigma^j \tag{36}$$

which are requisite for obtaining descriptions that are covariant with respect to local Poincaré transformations. For example, the quantities $\hat{g}_{ij}$ [defined by (14) in terms of the ϕ's] do not transform according to the indicated tensor law of transformation precisely because the ϕ's have affine laws of transformation, while the quantities g_{ij} [defined by (13) in terms of the B's] do have the indicated tensor law of transformation because the B's transform linear and the L's are Lorentz transformation matrices. It is also clear from (33) that it is impossible to construct P_{10}-invariant scalars from the translational torsion 2-forms except in those exceptional cases where all of the $L(4, \mathbb{R})$-curvature 2-forms θ^α vanish identically (i.e., the values of the $'\Omega$'s can be changed in any fashion we please by simply choosing the translation functions $t^i(x^k)$ in an appropriate fashion). Previous publications that purport to construct P_{10}-invariant free-field Lagrangians that only depend on the components of translational torsion are thus implicitly deceptive.

The representation of the holonomic curvature tensor of L_4 given by (28) and (29) is specific to the direct gauge theory of the Poincaré group. Previous gauge theories of P_{10} have used the standard representation of curvature in terms of Christoffel symbols, the components of the holonomic torsion tensor, and the components of the metricity tensor, in the now standard Schouten (1954) format. The curvature representations of previous theories thus involved second derivatives of $\hat{g}_{ij}$ and were nonlinear in first derivatives of $\hat{g}_{ij}$, because it is $\hat{g}_{ij}$ that is used as the metric tensor in most previous publications. The representation of the curvature tensor given by (28) and (29) uses only first derivatives of the compensating fields for the Lorentz sector (i.e., the W's). Further, it has only algebraic nonlinearities in the compensating fields, while the derivatives of the compensating fields occur only linearly. There is thus a significant and fundamental difference in the curvature representations used in direct gauge theory for the Poincaré group. As with most problems, it is hitting on the "right" representation that makes things work.

3. GAUGE-COVARIANT DERIVATIVES AND CONSTANT FIELDS

Now that we have determined the various connection 1-forms for the local action of the Poincaré group, it is an easy matter to define a gauge-covariant derivative; simply successively strange each living index with a similar index on the coefficients of the corresponding connection 1-forms. For instance, the ordinary covariant derivatives of the fundamental coframe fields B^i (considered as covectors on L_4) are given by

$$\nabla_k B^i_j = \partial_k B^i_j - \Gamma^m_{kj} B^i_m \tag{37}$$

In contrast, gauge-covariant derivatives have the evaluations

$$\overset{P}{\nabla}_k B^i_j = \partial_k B^i_j - \Gamma^m_{kj} B^i_m + \gamma^i_{km} B^m_j \tag{38}$$

$$\overset{P}{\nabla}_k L^i_{\alpha j} = \partial_k L^i_{\alpha j} + \Gamma^i_{km} L^m_{\alpha j} - \Gamma^m_{kj} L^i_{\alpha m} - \Gamma^\beta_{k\alpha} L^i_{\beta j} \tag{39}$$

$$\overset{P}{\nabla}_k l^i_{\alpha j} = \partial_k l^i_{\alpha j} + \gamma^i_{km} l^m_{\alpha j} - \gamma^m_{kj} l^i_{\alpha m} - \Gamma^\beta_{k\alpha} l^i_{\beta j} \tag{40}$$

When we substitute the explicit evaluations (19), (20), and (24) of the connection coefficients into three equations, we obtain

$$\overset{P}{\nabla}_k B^i_j = 0, \qquad \overset{P}{\nabla}_k L^i_{\alpha j} = 0, \qquad \overset{P}{\nabla}_k l^i_{\alpha j} = 0 \tag{41}$$

for all choices of the compensating fields for the local action of P_{10}. The fields B^i_j, $L^i_{\alpha j}$, and $l^i_{\alpha j}$ are therefore *gauge covariant constant* fields on L_4.

These results are special cases of a general result established in Edelen (1986*b*): *A gauge covaraint constant field on L_4 is the left to L_4 of a Poincaré invariant field on Minkowski space-time by minimal replacement.* Now, the metric tensor g_{ij} on L_4 that is given by (13) is the lift of the Poincaré invariant metric tensor h_{ij} on M_4 by minimal replacement. We therefore have

$$\overset{P}{\nabla}_k g_{ij} = \nabla_k g_{ij} = 0 \tag{42}$$

since ordinary covariant differentiation and gauge-covariant differentiation agree when applied to ordinary tensor fields. This result shows that the space L_4 is a *Riemann–Cartan* space U_4, without further assumptions or conditions. Previous theories use $\hat{g}_{ij}$ as metric tensor. These theories thus have to impose the "metric compatibility conditions" $\nabla_k \hat{g}_{ij} = 0$ in order to determine the connection coefficients. There is a particularly strong contrast here, for we have shown that direct gauge theory for P_{10} has uniquely determined connection coefficients and gives metric compatibility (satisfaction of the Ricci lemma) as an explicit consequence of minimal replacement, while previous theories have to put metric compability in by hand.

The implications of direct minimal replacement for P_{10} are more extensive than just securing automatic satisfaction of metric compatbility. We have already established the curvature representation

$$R^{i}_{rsj} = \theta^{\alpha}_{rs} L^{i}_{\alpha j}$$

and the fact that the $\mathbf{L}_\alpha$ form a basis for the Lie algebra of the Lorentz group on U_4 ($\equiv L_4$). Since the $\mathbf{L}_\alpha$ are now known to be gauge-covariant constant [see (39)], the known properties of the local holonomy group of a Riemann–Cartan space-time (Schouten, 1954; Hlavatý, 1959) show that *the local holonomy group of U_4 is the component of the Lorentz group connected to the identity.* Accordingly, parallel transport of geometric object fields around sufficiently small closed paths in U_4 is compatible with the kinematics of the tangent Minkowski space-time.

Simple and direct procedures are available for the representation of spin structures on U_4. Let $\{\gamma^i \mid 1 \leq i \leq 4\}$ be a set of generators for the Dirac algebra on M_4. This basis lifts to U_4 by minimal replacement to give

$$\sigma^i = b^i_k \gamma^k \tag{43}$$

(i.e., simply note that $\mathcal{M}\langle \gamma^i \partial_i \rangle = \sigma^i \partial_i$). It is then an easy matter to see that $\{\sigma^i \mid 1 \leq i \leq 4\}$ is a set of generators for the Dirac algebra on U_4 because

$$\sigma^i \sigma^j + \sigma^j \sigma^i = 2g^{ij} \tag{44}$$

when we use (6) and (13) to obtain $g^{ij} = b^i_r h^{rs} b^j_s$. If Ψ is a 4-component spinor field on Minkowski space-time, infinitesimal Lorentz transformations induce the transition

$$\Psi \to \Psi + \Delta u^\alpha M_\alpha \Psi + o(\Delta u^\beta) \tag{45}$$

where the Δu's are canonical coordinates of the group space of $L(4, \mathbb{R})$ in a sufficiently small neighborhood of the identity. The M's in (45) constitute a representation of $L(4, \mathbb{R})$ on the space of spinors on M_4, so they are determined by

$$\gamma^k M_\alpha - M_\alpha \gamma^k = l^k_{\alpha m} \gamma^m$$

that is

$$8M_\alpha = l^p_{\alpha v} h_{pu} (\gamma^u \gamma^v - \gamma^v \gamma^u) = L^p_{\alpha v} g_{pu} (\sigma^u \sigma^v - \sigma_v \sigma^u) \tag{46}$$

If we use $K_i\, dx^i$ to denote the spin connection 1-forms on U_4, an easy calculation based on (45) gives

$$K_i = W^\alpha_i M_\alpha = \tfrac{1}{8} W^\alpha_i L^p_{\alpha v} g_{pu} (\sigma^u \sigma^v - \sigma^v \sigma^u) \tag{47}$$

and hence

$$\overset{P}{\nabla}_k \sigma^i = \partial_k \sigma^i + K_k \sigma^i - \sigma^i K_k + \Gamma^i_{kj} \sigma^j = 0 \tag{48}$$

We thus see that *the generators of the Dirac algebra that are lifted to U_4 by minimal replacement are gauge-covariant constant fields on U_4.*

4. MATTER FIELDS AND LAGRANGIANS

Minimal replacement for the matter fields is given by $\mathcal{M}\langle\Psi\rangle = \Psi$, but derivatives of the matter fields are another story. Since the total group is $P_{10} \otimes G_r$, transformations in a sufficiently small neighborhood of the identity induce the transitions

$$\Psi \to \Psi + \Delta u^\alpha M_\alpha \Psi + \Delta v^a f_a \Psi + o(\Delta u^\beta, \Delta v^b) \tag{49}$$

where the M's form a representation of the Lie algebra of the Lorentz group on the matter fields and the f's form a representation of the Lie algebra of the internal symmetry group G_r. This is the usual situation, although there is no difficulty if the matter fields also provide a representation of the Lie algebra of the translation subgroup (Edelen, 1984, 1985*a*). Minimal replacement thus gives

$$\mathcal{M}\langle d\Psi\rangle = (\partial_k \Psi + W_k^\alpha M_\alpha \Psi + A_k^a f_a \Psi)\, dx^k \tag{50}$$

where $\{A^a = A_k^a\, dx^k,\ 1 \le a \le r\}$ are the compensating 1-forms for the local action of the internal symmetry group G_r. On the other hand, $d\Psi = (\partial_i \Psi)\, dx^i$, and the simultaneous action of P_{10} on the matter fields and on the base manifold yield

$$\mathcal{M}\langle d\Psi\rangle = \mathcal{M}\langle \partial_i \Psi\rangle \mathcal{M}\langle dx^i\rangle = y_i B_j^i\, dx^j \tag{51}$$

A combination of (50) and (51) thus gives

$$\begin{aligned}\mathcal{M}\langle \partial_i \Psi\rangle = y_i &= b_i^k(\partial_k \Psi + W_k^\alpha M_\alpha \Psi + A_k^a f_a \Psi)\\ &= b_i^k(\overset{P}{\nabla}_k \Psi + A_k^a f_a \Psi)\end{aligned} \tag{52}$$

The quantities inside the parenthesis are $P_{10} \otimes G_r$ gauge-covariant derivatives on U_4, while the contractions with the b's convert them to the corresponding anholonomic representations that always result when quantities are lefted from M_4 to U_4 by minimal replacement.

The explicit Yang–Mills gauge constructs start with the Lagrangian 4-form $\mathcal{L}_0(\Psi, \partial_i \Psi)\mu$ of the matter fields. Minimal replacement gives the new Lagrangian 4-form

$$\mathcal{L}_1 \mu = \mathcal{M}\langle \mathcal{L}_0 \mu\rangle = \mathcal{L}_0(\Psi, y_i)(-g)^{1/2}\mu \tag{53}$$

because $\mathcal{M}\langle\mu\rangle = B\mu = (-g)^{1/2}\mu$. Thus, if $\mathcal{L}_0$ is the "standard" Lagrangian

$$\mathcal{L}_0 = \bar{\Psi}(\gamma^j i\, \partial_j \Psi - m\Psi)$$

we obtain

$$\mathcal{L}_1 = \bar{\Psi}\{\sigma^j i(\overset{P}{\nabla}_j \Psi + A_j^a f_a \Psi) - m\Psi\}(-g)^{1/2}$$

The emergence of the gauge-covariant constant generators σ^j comes about from the contractions of the γ^i with the factors b_i^j in the expressions for the y's given by (52). In particular, we have the gauge-covariant coupling terms $\sigma^j A_j^a f_a \Psi$ rather than $\gamma^j A_j^a f_a \Psi$, and hence $\mathcal{L}_1$ is invariant under local action of the total group $P_{10} \otimes G_r$.

This is only part of the story, because the total Lagrangian is obtained by a Yang–Mills minimal coupling construction:

$$\mathcal{L} = \mathcal{L}_1 + (-g)^{1/2}\mathcal{V} \tag{54}$$

where $\mathcal{V}$ is a $(P_{10} \otimes G_r)$-invariant Lagrangian that depends on all of the compensating fields and their derivatives, but not on the matter fields nor on this derivatives. It has been shown (Edelen, 1985d) that a restriction to terms of algebraic degree two or less leads to the decomposition

$$\mathcal{V} = \mathcal{V}_P(\Sigma_{jk}^i, \theta_{jk}^\alpha, B_j^i) + \mathcal{V}_G(F_{jk}^a, B_j^i) \tag{55}$$

where the F's are the components of the curvature 2-forms for the internal symmetry group G_r.

Previous gauge theories of the Poincaré group have capitalized on the dependence of $\mathcal{V}_P$ on the torsion components, although the torsion used is usually the translational torsion Ω^i rather than the Cartan torsion. Now, it is questionable whether a P_{10}-invariant scalar can be constructed from the Ω's, because of the occurrence of the arbitrary position-dependent translation functions on the right-hand sides of (33). The Cartan torsion does not suffer from this difficulty, because it transforms homogeneously. On the other hand, we know that Σ^i has the evaluation

$$\Sigma^i = \theta^\alpha l_{\alpha j}^i x^j + \Omega^i$$

and hence a dependence of $\mathcal{V}_P$ on the Cartan torsion introduces explicit dependence on the independent variables. This explicit coordinate dependence will destroy the local conservation laws that obtain from global translation invariance. Further, it has been shown (Edelen, 1985a–d) that dependence of the Lagrangian on Cartan torsion gives rise to physically unreasonable contributions to the spin current 3-forms.

If we start with the differential system $D\Psi = \mathcal{M}\langle d\Psi\rangle$ in standard Yang–Mills theory for a semisimple symmetry group, the gauge torsion (Edelen, 1985e, Chapter 5) generated by this differential system has the evaluation $DD\Psi = F\Psi$, with F = curvature 2-forms of the internal symmetry group. Yang–Mills minimal coupling thus precludes a dependence of the "free gauge field" Lagrangian on the gauge torsion. For the Poincaré group, we start with the differential system $B^i = \mathcal{M}\langle dx^i \rangle$ and obtain the Cartan torsion by $DB^i = \Sigma^i$ with the explicit dependence on the independent variables x^i shown above. Since such an explicit dependence on the independent variables destroys conservation of the variationally defined momentum-energy complex (Edelen, 1985*e*, Chapter 7), we should likewise preclude dependence of the "free gauge field" Lagrangian $\mathcal{V}$ on the Cartan torsion. In fact, it is clear that an alternative formulation of the classical Yang–Mills minimal coupling construction is that "free gauge field" Lagrangians can depend on gauge curvature, but *not* on gauge torsion.

Situations that obtain under the restriction that $\mathcal{V}$ does not depend on the components of torsion have been analyzed by Edelen (1985*d*) under the additional assumption that $\mathcal{V}_P$ takes a form similar to the Einstein–Hilbert Lagrangian of general relativity:

$$\mathcal{V}_P = k_0 + k_1 R, \qquad R = R^r_{rij} g^{ij} \tag{56}$$

Although a dependence on quadratic and higher order terms in the components of the curvature tensor could be accommodated, they would lead to a violation of the Birkhoff property even if a method of determining the coefficients of such terms could be reduced to reasonable experimental questions.

We assume that $\mathcal{V}_G$ has been chosen so that the theory gives an adequate description of the physics on Minkowski space-time (i.e., with global action of the Poincaré group). The procedure given here provides a definite theory in the new Riemann–Cartan space-time U_4 that grows out of Minkowski space-time by minimal replacement and minimal coupling (Edelen, 1985*d*, 1986*a*). The following results are then obtained. Constants k_0 and k_1 that appear in (56) are uniquely determined by the cosmological constant and the general relativistic coupling constant $8\pi G/c^4$. There are thus no free constants that are allowed to "float" in value. Torsion is *algebraically* determined by the components of the spin 3-forms of the matter fields (it does not propagate). When the total momentum-energy complex is symmetric and the matter fields are spin-free, the theory reduces exactly to general relativity. In addition, it has been shown (Edelen, 1986*a*) that a homogeneous scaling of the generators of the Poincaré group provides a direct mechanism for regular asymptotic expansion in the gravitational coupling constant. This expansion gives (1) the Minkowski space-time formulation of the physics as leading terms, (2) a breaking of the local action of the Poincaré group down to global action, and (3) well-defined equations that determine the $O(8\pi G/c^4)$ (i.e., gravitational) corrections to the Minkowski space-time formulation for the matter fields, the compensating fields for local action of the internal symmetry group, and the compensating fields for local action of the Poincaré group. This is similar to classical symmetry breaking, although we expand about dynamic equilibrium states (the Minkowski space-time formulation) rather than about static equilibrium states (local minima of the potential energy). There is thus a reasonable expectation that direct gauging of the Poincaré group will provide a mechanism for "turning on gravity" for any model that is correctly formulated at the Minkowski level, and this happens without the inconsistent and spurious results prevalent in previous formulations.

5. TEST PARTICLES AND GRAVITATIONAL FORCES

We have seen that the space U_4 that grows out of Minkowski space-time by minimal replacement is a Riemann–Cartan space-time. It is therefore covered by the families of geodesic curves that provide preferred road maps by which we find our way around in U_4 in an unambiguous manner. Indeed, if we follow the original arguments laid down by Einstein and elaborated by Synge (1960), the geodesics in U_4 may be identified with the paths of electrically neutral test particles that serve to detect gravitational effects.

A geodesic in U_4 is a curve whose tangent vector is parallel translated along that curve. If we parametrize the curve with the arc length S along the curve from some convenient reference point by $x^i = U^i(S)$ and use an overdot to denote differentiation with respect to S, the equations for the geodesic curves are given by

$$\dot{U}^i = V^i, \qquad \dot{V}^i + \Gamma^i_{jk} V^j V^k = 0 \tag{57}$$

This form of the geodesic equations, although adequate, does not take advantage of the additional structure that is afforded by the construction of U_4 out of M_4 by minimal replacement. Since U_4 has canonical frame and coframe fields $\{b_i, B^i\}$, we may resolve the tangent vector field of the geodesic by

$$V^i = b^i_k v^k, \qquad v^i = B^i_j V^j \tag{58}$$

Noting that the frame and coframe fields on U_4 are gauge covariant constant, it follows that (57) goes over into the equivalent system [see (20)]

$$\dot{U}^i = b^i_k v^k, \qquad \dot{v}^i + V^j \gamma^i_{jk} v^k = 0 \tag{59}$$

We now use (19) and set

$$\omega^\alpha = V^j W^\alpha_j = V \lrcorner W^\alpha = v^k b^j_k W^\alpha_j \tag{60}$$

in which case the system (59) takes the equivalent form

$$\dot{U}^i = b^i_k v^k, \qquad \dot{v}^i + \omega^\alpha l^i_{\alpha k} v^k = 0 \tag{61}$$

It is an easy matter to see that (61) admits the quadratic first integral

$$v^i h_{ij} v^j = k = \text{const} \tag{62}$$

If we let the first three l's be a basis for the subgroup $SO(3, \mathbb{R})$, the remaining three l's generate infinitesimal Lorentz boosts in the (x, t), (y, t), and (z, t) planes (Edelen, 1986*b*), and if we choose units so that the speed of light is unity, the second half of the system (61) becomes

$$\dot{v}^1 + \omega^1 v^2 + \omega^2 v^3 + \omega^4 v^4 = 0, \tag{63}$$

$$\dot{v}^2 - \omega^1 v^1 + \omega^3 v^3 + \omega^5 v^4 = 0 \tag{64}$$

$$\dot{v}^3 - \omega^2 v^1 - \omega^3 v^2 + \omega^6 v^4 = 0 \tag{65}$$

$$\dot{v}^4 + \omega^4 v^1 + \omega^5 v^2 + \omega^6 v^3 = 0 \tag{66}$$

Inspection of these equations shows that ω^1, ω^2, and ω^3 generate rotations in the 3-dimensional space of the tangent space that is spanned by v^1, v^2, and v^3. Hence the first three ω's are associated with intrinsic angular accelerations that come from the compensating 1-forms for local action of $SO(2, \mathbb{R})$ [see (60)]. On the other hand, we see that ω^4, ω^5, and ω^6 generate changes in the magnitudes of v^1, v^2, v^3 that are consistent with the first integral (62). The latter three ω's are thus associated with linear accelerations. This is corroborated by (66), which takes the form of an "energy" equation. Now, the last three ω's come from the compensating 1-forms for the three Lorentz boosts [see (60)]. This shows that it is reasonable to associate the compensating 1-forms for local Lorentz boots with gravitational effects. Indeed, the Lorentz boosts are the Minkowski-space generalization of the classical Newtonian symmetries associated with the motion of the center of mass. As such, the U_4 generalization of Lorentz boosts should reflect changes in the center of mass of the test particle that occur as a consequence of the nontrivial geometric structure of U_4.

Previous gauge theories for the Poincaré group have associated gravitational effects with the local action of the translation subgroup. Now that we know that this interpretation is inconsistent with the equations of motion of electrically neutral test particles, the question naturally arises as to how to interpret the effects of local action of the translation subgroup. We have seen that local action of the translation subgroup leads to general covariance of the theory in U_4 under abritrary smooth changes of coordinate covers, even though the Minkowski space formulation is only covariant under global Lorentz transformations. This shows that local action of the translation subgroup is essential in order to obtain a generally covaraint field theory. The question thus remains as to how we should interpret the compensating 1-forms for the translation subgroup. A similar question emerges with respect to the rotation subgroup $SO(3, \mathbb{R})$.

A bold step can be taken by using the known properties of gauge theories based on $SO(3) \rhd T(3)$ as an internal symmetry group for classical solid bodies (Kadić and Edelen, 1983; Edelen and Lagoudas, 1988). These results suggest that the compensating 1-forms for the translation subgroup describe *dislocation* degrees of freedom of space-time. Similar arguments would allow us to interpret the compensating 1-forms for the subgroup $SO(3)$ as describing *desclination* degrees of freedom of space-time. Unfortunately, a clear physical meaning of such space-time dislocations and disclinations has yet to appear. A possible explanation could be that space-time dislocations and disclinations require very large activation energies, since they do not appear to be relevant at the low- and moderate-energy regions of current experiments. An alternative conjecture would be that they have not been detected because no one has looked for them through explicitly constructed experiments. It is clear, however, that the ten 1-form degrees of freedom associated with local action of the Poincaré group contain more information than is needed in order to model gravitational interactions. This additional information should reflect possible physical degrees of freedom of the fundamental fields if we are to believe that the Poincaré group is the fundamental group at the tangent space approximation. These ideas open new vistas in fundamental field theory, and indicate an inherent richness that has yet to be fully probed.

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